Navigation of Quantum-Controlled Mobile Robots

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1. Introduction

The actions of autonomous mobile robots in stochastic medium imply certain intellectual behavior, which allows fulfilling the mission in spite of the environmental uncertainty and the robot's influence on the characteristics of the medium. To provide such a behavior, the controllers of the robots are considered as probabilistic automata with decision-making and, in some cases, learning abilities. General studies in this direction began in the 1960s (Fu & Li, 1969; Tsetlin, 1973) and resulted in practical methods of on-line decision-making and navigation of mobile robots (Unsal, 1998; Kagan & Ben-Gal, 2008).

Along with the indicated studies, in recent years, the methods of mobile robot's navigation and control are considered in the framework of quantum computation (Nielsen & Chuang, 2000), which gave rise to the concept of quantum mobile robot (Benioff, 1998; Dong, et al., 2006). Such approach allowed including both an environmental influence on the robot's actions and the changes of the environment by the robot by the use of the same model, and the ability to apply the methods of quantum communication and decision-making (Levitin, 1969; Helstrom, 1976; Davies, 1978; Holevo, 2001) to the mobile robot's control.

Following Benioff, *quantum robots* are "mobile systems that have a quantum computer and any other needed ancillary systems on board… Quantum robots move in and interact (locally) with environments of quantum systems" (Benioff, 1998). If, in contrast, the quantum robots interact with a non-quantum environment, then they are considered as *quantum-controlled mobile robots*. According to Perkowski, these robots are such that "their controls are quantum but sensors and effectors are classical" (Raghuvanshi, et al., 2007). In the other words, in the quantum-controlled mobile robot, the input data obtained by classical (non-quantum) sensors are processed by the use of quantum-mechanical methods, and the results are output to classical (non-quantum) effectors.

In this chapter, we present a brief practical introduction into quantum computation and information theory and consider the methods of path planning and navigation of quantum-controlled mobile robots based on quantum decision-making.

2. Quantum information theory and models of quantum computations

We begin with a brief introduction into quantum information theory and stress the facts, which are required for the tasks of mobile robots' navigation. An application of quantum-mechanical methods for the mobile robot's control is based on the statistical interpretation of

quantum mechanics (Ballentine, 2006); for a complete review of quantum computation and information theory, see, e.g., (Nielsen & Chuang, 2000).

2.1 Basic notation and properties of quantum-mechanical systems

In general, in the considerations of the finite quantum-mechanical systems, it is postulated (Ballentine, 2006) that the state of the quantum-mechanical system is represented by a Hermitian complex matrix $\sigma = \left\| \sigma_{jk} \right\|$, $\sigma_{jk} = \sigma_{kj}^{*}$, where $(\cdot)^{*}$ stands for complex conjugate, with the unit sum of the diagonal elements, $\mathrm{tr}(\sigma) = 1$; and that the observed value of the quantum-mechanical system is specified by the eigenvalues of the matrix σ. Since matrix σ is Hermitial, its eigenvalues are real numbers. If matrix σ is diagonal, then such a representation of the state is equivalent to the representation of the state of stochastic classical system, in which diagonal elements σ_{jj} form a probability vector (Holevo, 2001).

Let $\bar{s} = (s_1, s_2, \ldots)$ be a vector of complex variables $s_j = a_j + ib_j$, $j = 1, 2, \ldots$, and let $s_j^{*} = a_j - ib_j$ be its complex conjugate. According to the Dirac's "bra[c]ket" notation, row-vector $\langle s | = (s_1, s_2, \ldots)$ is called *bra-vector* and a column-vector $| s \rangle = \left(s_1^{*}, s_2^{*}, \ldots \right)^{T}$, where $(\cdot)^{T}$ stands for the transposition of the vector, is called *ket-vector*. Vector $\bar{s}$ or, equivalently, a Hermitian matrix $\rho = | s \rangle \langle s |$, are interpreted as a state of quantum object; vector $\bar{s}$ is called *state vector* and matrix ρ is called *state matrix* or *density matrix*.

Let $\bar{s}_1 = (s_{11}, s_{12}, \ldots)$ and $\bar{s}_2 = (s_{21}, s_{22}, \ldots)$ be two state vectors. The inner product $\langle s_2 | s_1 \rangle = (s_{21}, s_{22}, \ldots) \cdot \left(s_{11}^{*}, s_{12}^{*}, \ldots \right)^{T} = s_{21} \cdot s_{11}^{*} + s_{22} \cdot s_{12}^{*} + \ldots$ is called *amplitude* of the event that the object moves from the state $\bar{s}_1$ to the state $\bar{s}_2$. The *probability* of this event is defined by a square of the absolute value of the amplitude $\langle s_2 | s_1 \rangle$, i.e. $P(s_2 | s_1) = \left| \langle s_2 | s_1 \rangle \right|^2$, where $\left| \langle s_2 | s_1 \rangle \right| = |\alpha + i\beta| = \sqrt{\alpha^2 + \beta^2}$. By definition, regarding the probability of any state $\bar{s}$, it is assumed that $0 \le P(s | s) \le 1$. For example, let $\langle s_1 | = (1, 0)$ and $\langle s_2 | = \left(1/\sqrt{2}, 1/\sqrt{2} \right)$, where both $P(s_1 | s_1) = 1$ and $P(s_2 | s_2) = 1$. However, the probability of transition from the state $\bar{s}_1$ to the state $\bar{s}_2$ is $P(s_2 | s_1) = \left| \langle s_2 | s_1 \rangle \right|^2 = 1/2$.

The *measurement* of the state of the object is described by an *observation operator* O, which in the finite case is determined by a squire matrix $O = \left\| o_{jk} \right\|$. The result of the observation of the state $\bar{s}$ is defined by a multiplication $\langle s | O | s \rangle = (s_1, s_2, \ldots) \cdot \left\| o_{jk} \right\| \cdot \left(s_1^{*}, s_2^{*}, \ldots \right)^{T}$, and the state, which is obtained after an observation is defined as $| s' \rangle = O | s \rangle$. For example, assume that the measurement is conducted by the use of the operator $O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, the observation of the states $\langle s_1 | = (1, 0)$ and $\langle s_2 | = \left(1/\sqrt{2}, 1/\sqrt{2} \right)$ results in $\langle s_1 | O | s_1 \rangle = 1$ and $\langle s_2 | O | s_2 \rangle = 0$, i.e. operator O unambiguously detects the states $\bar{s}_1$ and $\bar{s}_2$. Moreover, an observation of the state $\bar{s}_1$ does not change this state, i.e. $O | s_1 \rangle = (1, 0)^{T}$, while the observation of the state $\bar{s}_2$ results in the new state $O | s_2 \rangle = \left(1/\sqrt{2}, -1/\sqrt{2} \right)^{T}$. From such a property of observation, it follows that in contrast to the classical systems, the actual state of the quantum-mechanical system obtains a value, which was measured by the observer, and further evolution of the system starts from this value. In the other words, the evolution the quantum-mechanical system depends on the fact of its observation.

An actual evolution of the quantum-mechanical system is governed by the evolution operators, which are applied to the state matrix σ or state vector $\bar{s}$. Below, we consider the states and operators, which are used in quantum information theory.

2.2 Concepts of the quantum information theory

The elementary state, which is considered in quantum information theory (Nielsen & Chuang, 2000), is called *qubit* (quantum bit) and is represented by a two-element complex vector $|s\rangle = (s_1, s_2)^T$, $0 \le |s_1|^2 + |s_2|^2 \le 1$. Among such vectors, two vectors are specified $|0\rangle = (1, 0)^T$ and $|1\rangle = (0, 1)^T$, which correspond to the bit values "0" and "1" known in classical information theory (Cover & Thomas, 1991). In general, vectors $|0\rangle$ and $|1\rangle$ determine the states "spin up" and "spin down" of electron, i.e. $|0\rangle \equiv "\uparrow"$ and $|1\rangle \equiv "\downarrow"$.

Given vectors $|0\rangle = (1, 0)^T$ and $|1\rangle = (0, 1)^T$, any qubit $|s\rangle = (s_1, s_2)^T$ is represented as $|s\rangle = (s_1, s_2)^T = s_1|0\rangle + s_2|1\rangle$. In particular, if there are defined two states $|"\rightarrow"\rangle = (1/\sqrt{2}, 1/\sqrt{2})^T$ and $|"\leftarrow"\rangle = (1/\sqrt{2}, -1/\sqrt{2})^T$, which represent the electron states "spin right" and "spin left", then $|"\rightarrow"\rangle = (1/\sqrt{2})\cdot|1\rangle + (1/\sqrt{2})\cdot|0\rangle$ and $|"\leftarrow"\rangle = (1/\sqrt{2})\cdot|1\rangle - (1/\sqrt{2})\cdot|0\rangle$. Moreover, since it holds true that $|0\rangle = (1/\sqrt{2})\cdot|"\rightarrow"\rangle + (1/\sqrt{2})\cdot|"\leftarrow"\rangle$ and $|1\rangle = (1/\sqrt{2})\cdot|"\rightarrow"\rangle - (1/\sqrt{2})\cdot|"\leftarrow"\rangle$, the pairs of the vectors $|0\rangle$ and $|1\rangle$, and $|"\rightarrow"\rangle$ and $|"\leftarrow"\rangle$ can be used interchangeably.
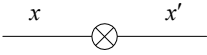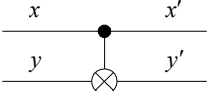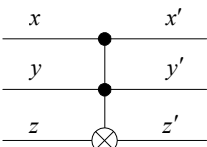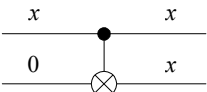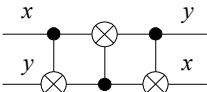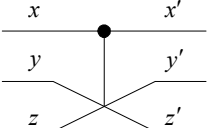
In general, the evolution of the qubits is governed by the use of the following operators:

Pauli operators: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

- Hadamard operator: $H = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, Phase shift operator: $S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$;

- Controlled NOT (CNOT) operator: $CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$.

The Pauli operators are the most known qubits operators that are in use in general quantum mechanics, while the other three operators are more specific for quantum information theory. According to the Kitaev-Solovey theorem (Nielsen & Chuang, 2000), an algebra $\cup = \{\{|0\rangle, |1\rangle\}, CNOT, H, S\}$, which consists of the qubits $|0\rangle$ and $|1\rangle$, and CNOT, Hadamard and phase shift operators, forms a universal algebra that models any operation of the Boolean algebra $\mathbb{B} = \{\{0, 1\}, \neg, \&, \vee\}$. Notice that the qubit operators are reversible. In fact, direct calculations show that $NOT(NOT(\bar{s})) = X\cdot X\cdot|s\rangle = |s\rangle$, $H\cdot H\cdot|s\rangle = |s\rangle$ and so on.

To illustrate the actions of the simplest qubit operators and their relation with classical logic operations, let us present the corresponding quantum gates and their pseudocode.

NOT ($x' := \bar{x}$): **if** $x = 1$ **then** $x' = 0$ **else** $x' = 1$ $x \quad x'$	CNOT: $x' = x$ **if** $x = 1$ **then** $y' = \bar{y}$ **else** $y' = y$ $x \quad x'$ $y \quad y'$
CCNOT (TOFFOLI GATE): $x' = x$, $y' = y$ **if** $x = 1$ **and** $y = 1$ **then** $z' = \bar{z}$ **else** $z' = z$ $x \quad x'$ $y \quad y'$ $z \quad z'$	FAN OUT: CNOT with constant input $y := 0$ $x \quad x$ $0 \quad x$
EXCHANGE: $x' := y$, $y' := x$ $x \quad y$ $y \quad x$	C-EXCHANGE (FREDKIN GATE): $x \quad x'$ $y \quad y'$ $z \quad z'$

The other types of the qubit gates, e.g. phase shift operator S and its derivatives, cannot be represented by the classical operators and require quantum computing devices. In such computing, it is assumed that each matrix operation is conducted in one computation step providing a power of quantum computing. The indicated dependence of quantum states on the observation process allows an implementation of such operations by the use of adaptive computation schemes. Below, we demonstrate a relation between quantum operations and evolving algebras and consider appropriate probabilistic decision-making.

3. Probabilistic automata and mobile robots control

In this section we describe a simulation example, which illustrates an implementation of qubit operators for the mobile robot's navigation. Then, based on this example, we consider the model of probabilistic control and algorithmic learning methods based on evolving algebras.

3.1 Control of the mobile robot by qubit operators

Let us consider a mobile robot, which moves on the plain, and assume that the inner states of the robots correspond to its direction on a plane. Let $S = \{ "\uparrow", "\downarrow", "\rightarrow", "\leftarrow" \}$ be a set of pure or errorless states such that each state corresponds to a certain direction of the robot. The set $V = \{ "step\ forward", "step\ backward", "turn\ left", "turn\ right", "stay\ still" \}$ includes the actions that are available to the robot, where the steps "*forward*" and "*backward*" are restricted by a certain fixed distance. Being at time moment t in the state $s^t \in S$ and choosing the action $v^t \in V$, the robot receives to its input a finite scalar value $\varepsilon^t = \varepsilon\left(s^t, v^t\right)$, which depends on the robot's position in the environment. Notice that the elements of the set V form a group with multiplication acting on the set of states S . The steps "*forward*", "*backward*" and "*stay still*" action do not change the state of the robot but change its position relatively to the environment that changes the obtained value ε .

Assume that the states of the robot are described by four qubits: $"\uparrow" = |s_1\rangle = (1, 0)^T$, $"\downarrow" = |s_2\rangle = (0, 1)^T$, $"\rightarrow" = |s_3\rangle = (1/\sqrt{2}, 1/\sqrt{2})^T$ and $"\leftarrow" = |s_4\rangle = (1/\sqrt{2}, -1/\sqrt{2})^T$, and the actions "*turn right*" and "*turn left*" are, respectively, conducted by the use of the Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and reverse Hadamard $H^R = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ operators. Then for the Hadamard operator it holds true that $H|s_1\rangle = |s_3\rangle$, $H|s_3\rangle = |s_1\rangle$, $H|s_2\rangle = |s_4\rangle$, and $H|s_4\rangle = |s_2\rangle$; and similarly, for the reverse Hadamard it holds true that $H^R|s_1\rangle = |s_4\rangle$, $H^R|s_4\rangle = |s_1\rangle$, $H^R|s_2\rangle = |s_3\rangle$ and $H^R|s_3\rangle = |s_2\rangle$, where the states $|s\rangle = a|0\rangle + b|1\rangle$ and $-|s\rangle = (-1)(a, b)^T$ conventionally are not distinguished.

Let us consider the states and the actions of the robot in particular. If there is no influence on the environment, then the relations between the states can be represented as follows. Assume that the robot is in the state $|s_1\rangle$. Then according to the above-indicated equality $|s_1\rangle = (1/\sqrt{2}) \cdot |s_3\rangle + (1/\sqrt{2}) \cdot |s_4\rangle$, and the probability that at the next time moment the robot will be in the state $|s_3\rangle$ is $\rho(s_3 | s_1) = \Pr\{s^{t+1} = |s_3\rangle | s^t = |s_1\rangle\} = |1/\sqrt{2}|^2 = 1/2$ and the probability that it will be at the state $|s_4\rangle$ is also $\rho(s_4 | s_1) = \Pr\{s^{t+1} = |s_4\rangle | s^t = |s_1\rangle\} = |1/\sqrt{2}|^2 = 1/2$. The similar equalities hold true for the other remaining states $|s_2\rangle$, $|s_3\rangle$ and $|s_4\rangle$. In general, for complex amplitudes s_{jk}, it follows that $|s_1\rangle = s_{13}|s_3\rangle + s_{14}|s_4\rangle$, $|s_2\rangle = s_{23}|s_3\rangle - s_{24}|s_4\rangle$, $|s_3\rangle = s_{31}|s_1\rangle + s_{32}|s_2\rangle$, and $|s_4\rangle = s_{41}|s_1\rangle - s_{42}|s_2\rangle$; thus $\rho(s_k | s_j) = \Pr\{s^{t+1} = |s_k\rangle | s^t = |s_j\rangle\} = |s_{jk}|^2$.

Now let us take into account an influence of the environment. Recall that the quantum state of the qubit is equivalently defined both by the state-vector $|s_i\rangle$ and by the density matrix. Let $|s_1\rangle = (1, 0)^T$ and $|s_2\rangle = (0, 1)^T$, and let $|s_3\rangle = (s_{31}, s_{32})^T$, $|s_4\rangle = (s_{41}, s_{42})^T$. Then, by definition, the states are defined by the density matrices $\sigma_1 = |s_1\rangle\langle s_1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma_2 = |s_2\rangle\langle s_2| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_3 = |s_3\rangle\langle s_3| = \begin{pmatrix} |s_{31}|^2 & s_{31}s_{32}^* \\ s_{31}^* s_{32} & |s_{32}|^2 \end{pmatrix}$, and $\sigma_4 = |s_4\rangle\langle s_4| = \begin{pmatrix} |s_{41}|^2 & s_{41}s_{42}^* \\ s_{41}^* s_{42} & |s_{42}|^2 \end{pmatrix}$, while $\operatorname{tr}(\sigma_i) = 1$. Since the non-diagonal elements of the matrices can obtain arbitrary values, let us use these elements for specifying the relation with the environmental variable ε. In particular, assume that $\sigma_3(\varepsilon) = \sigma_3 + \begin{pmatrix} 0 & i \cdot \varepsilon \\ -i \cdot \varepsilon & 0 \end{pmatrix}$. Then, the application of the Hadamard operators H and H^R to the state $\sigma_3(\varepsilon)$ after normalization results in the states $|s'\rangle = (1/\sqrt{2} - i(1/\sqrt{2})\varepsilon, i(1/\sqrt{2})\varepsilon)^T$ and $|s''\rangle = (-i(1/\sqrt{2})\varepsilon, 1/\sqrt{2} + i(1/\sqrt{2})\varepsilon)^T$. If, e.g., $\varepsilon = 1/3$, then $|s'\rangle = (0.72 - 0.24i, 0.24i)^T$ and

$\left| s'' \right\rangle = \left(-0.24i, 0.72 + 0.24i \right)^T$ instead of the states $\left| s_1 \right\rangle = \left(1, 0 \right)^T$ and $\left| s_2 \right\rangle = \left(0, 1 \right)^T$, which are obtained in the errorless case. The transition probabilities are $\rho \left(s_1 \mid s' \right) = \rho \left(s_2 \mid s'' \right) = 0.76$ and $\rho \left(s_2 \mid s' \right) = \rho \left(s_1 \mid s'' \right) = 0.24$. The presented method of control is illustrated by the example, shown in Fig. 1. In the simulation (Rybalov et al., 2010), the robot was programmed to follow the trajectory by the use of a compass sensor, and its turns were controlled by Hadamard and reverse Hadamard operators.
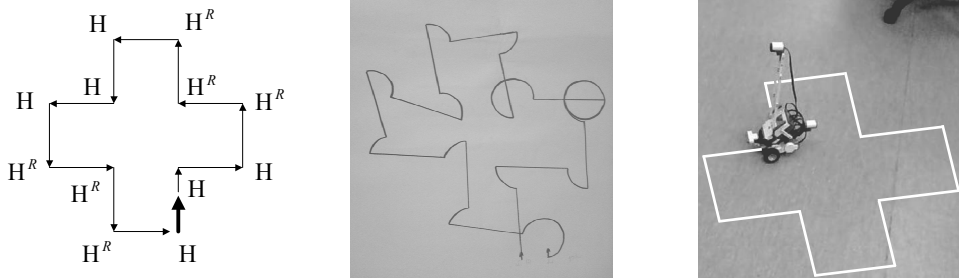

Fig. 1. From left to right: the planned trajectory with corresponding Hadamard and reverse Hadamard operator; the resulting trajectory with the directions of turns; and the mobile robot following the trajectory by the use of the compass sensor.

The presented approach to control the robot is a particular case of the dynamics of open quantum dynamical system (Holevo, 2001). In general, the dynamics of such systems is determined by transitions $\left| s \right\rangle \rightarrow \mathrm{U} \left| s \right\rangle$, where U is an appropriate Hermitian operator.

3.2 Probabilistic model and algorithmic learning

Let us take into account the observation process and present a brief description of the algorithmic model of the robot's control system with variable evolution structure. As it is usual for automata models, let X be a set of input values, Y be a set of output values, and S be a set of inner states. The automaton is defined by two characteristic functions $f : X \times S \rightarrow Y$ and $g : X \times S \rightarrow S$, such that discrete time dynamics is defined as $y^t = f \left(x^t, s^t \right)$ and $s^{t+1} = g \left(x^t, s^t \right)$, where $x^t \in X$, $y^t \in Y$, $s^t, s^{t+1} \in S$, $t = 0, 1, 2, \dots$ In the case of *learning automata*, the probability distributions are defined over the sets X, Y and S, and functions f and g act on such distributions (Fu & Li, 1969; Tsetlin, 1973), while the metrical or topological structures of the sets are constant. By the other approaches, the *program structure learning* is specified by a convergence to the appropriate transition function g, or by the choice of a metric or topology over the set S, called *data structure learning*.

Algorithmically, the variable data structure is defined as follows (Gurevich, 1991). Let G be a global namespace with three distinguished elements $1 = "true"$, $0 = "false"$ and $\lozenge = "undef"$. A map $\varphi : G^r \rightarrow G$ is called basic function of arity r, while a basic function of arity $r = 0$ is considered as a distinguished element, and basic functions $\psi : G^r \rightarrow \{1, 0, \lozenge\}$ are considered as terms. The evolving algebra $\mathbb{Y} = \left\{ G, \varphi_0, \varphi_1, \dots \right\}$ is defined by the following updates:

-   local update: $\varphi(a_0, a_1, \ldots, a_r) \leftarrow a$, which specifies the value to the basic function,

-   guarded update: if $\psi = 1$ then $\varphi(a_0, a_1, \ldots, a_r) \leftarrow a$,

where $a_0, a_1, \ldots, a_r, a \in G$ and ψ is a term. Thus, evolving algebra $\mathbb{V}$ permits its functions to change their domains according to the data flow, as it is required for the learning property. Notice that the implementation of the updates of the algebra $\mathbb{V}$ by the use of quantum gates results in the universal algebra $\maltese$, which defines quantum computations. Nevertheless, similar to the algebra $\maltese$, the direct implementation of the evolving algebra $\mathbb{V}$ is possible only in particular cases of computable operators.

Let us consider an implementation of the evolving algebra for probabilistic control of the mobile robots control (Kagan & Ben-Gal, 2008). Since during its mission the robot acts in a stochastic environment, the input variable $x \in X$ is random; thus, given characteristic f and g, both inner states $s \in S$ and outputs $y \in Y$ are also random variables. Assume that the namespace G, in addition to elements 1, 0 and $\Diamond$, includes all possible realizations of the inputs, inner states and outputs, i.e. $G = \{1, 0, \Diamond\} \cup X \cup Y \cup S$. For the realizations of inputs, we define the terms $\psi_1(x) \in \{1, 0, \Diamond\}$, and for the inner states we define the terms $\psi_2(s) \in \{1, 0, \Diamond\}$ and the local updates $\varphi(s') \leftarrow s''$, $s', s'' \in G$.

Now we can formulate transition function g in the terms of evolving algebra. We say that the pairs $(x', s'), (x'', s'') \in X \times S$ are equal (in the sense of the function g) if $g : (x', s') \mapsto s$ and $g : (x'', s'') \mapsto \bar{s}$. Then, since there are at most m distinguished realizations of the inner states and there may exist the pairs from $X \times S$, such that the map g is not defined, the space $X \times S$ is partitioned into at most $m + 1$ equivalence classes. Denote such equivalence classes by A^g. Then, the transition function g is defined as a guarded update if $(x, s) \in A_j^g$ then $\phi(s) \leftarrow s_j$, which is checked for each pair $(x, s) \in X \times S$, and if the appropriate class A^g is not found, then $\phi(s) \leftarrow \Diamond$ is specified.

The presented algorithmic model allows simulations of quantum-control of the mobile robot and its navigation on the basis of the qubits model of states. Below, we consider an example of such simulations with probabilistic decision-making.

4. Navigation of quantum-controlled mobile robot along predefined path

Let us start with a simulation example. The simulation follows an idea of the experiment of checking a spin of elementary particle by three Stern-Gerlach apparatus, which are defined by a sequence of certain quantum operators (Albert, 1994).

As above, let the inner states of the robot be specified by four qubits $"\uparrow" = |s_1\rangle = (1, 0)^T$, $"\downarrow" = |s_2\rangle = (0, 1)^T$, $"\rightarrow" = |s_3\rangle = \left(1/\sqrt{2}, 1/\sqrt{2}\right)^T$ and $"\leftarrow" = |s_4\rangle = \left(1/\sqrt{2}, -1/\sqrt{2}\right)^T$. In addition, assume that there are two types of detectors defined by the above-defined Pauli operators X and Z, so that $\langle s_3|X|s_3\rangle = +1$, $\langle s_4|X|s_4\rangle = -1$, $\langle s_1|Z|s_1\rangle = +1$ and $\langle s_2|Z|s_2\rangle = -1$. The robot starts with a random initial state $|s^0\rangle$ and arrives to the first detector Z. Detector Z checks the state $|s^0\rangle$ and the robot obtains a new state $|s^1\rangle$. According to the maximum of probabilities $P(s_3|s^1)$ and $P(s_4|s^1)$, the robot chooses the left or right trajectory and arrives

to the second detector X. Then, after the similar actions and obtaining the state $\left|s^2\right\rangle$, the robot continues to the third detector Z, which checks the robot's state and results in the state $\left|s^3\right\rangle$. The fragment of the experiment with the robot following its path is shown in Fig. 2.
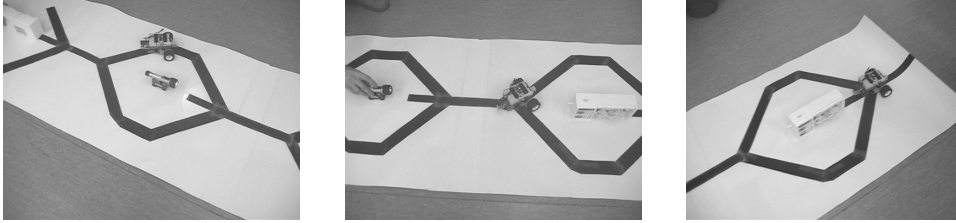

Fig. 2. The mobile robot follows a path (from right to left) with three simulated detectors using touch sensor (" Z detector") and light sensors (" X detector").

Let us consider the actions of the robot in particular. As indicated above, the robot starts with an initial state $\left|s^0\right\rangle = \left(s_1^0, s_2^0\right)^T$, where $s_1^0, s_2^0 \in [0, 1]$ are random values such that $0 < \left(s_1^0\right)^2 + \left(s_2^0\right)^2 \le 1$. Then the first detector Z results in the state $\left|s^1\right\rangle = Z\left|s^0\right\rangle$ and the decision-making is conducted regarding the further left or right trajectory, which is based on the probabilities $P\left(s_3 \mid s^1\right) = \left|\left\langle s^1 \middle| s_3\right\rangle\right|^2$ and $P\left(s_4 \mid s^1\right) = \left|\left\langle s^1 \middle| s_4\right\rangle\right|^2$. If $P\left(s_3 \mid s^1\right) > P\left(s_4 \mid s^1\right)$, then the robot turns left, and if $P\left(s_3 \mid s^1\right) < P\left(s_4 \mid s^1\right)$, then the robot turns right (the ties are broken randomly). Following the chosen trajectory, the robot arrives to the second detector X. The check with this detector results in the state $\left|s^2\right\rangle = X\left|s^1\right\rangle$, and the decision regarding the further trajectory is obtained on the basis of the probabilities $P\left(s_1 \mid s^2\right) = \left|\left\langle s^2 \middle| s_1\right\rangle\right|^2$ and $P\left(s_2 \mid s^2\right) = \left|\left\langle s^2 \middle| s_2\right\rangle\right|^2$. Similar to the above, if $P\left(s_1 \mid s^2\right) > P\left(s_2 \mid s^2\right)$, then the robot turns left, and if $P\left(s_1 \mid s^2\right) < P\left(s_2 \mid s^2\right)$, then the robot turns right (the ties are broken randomly). The third check is again conducted by the Z detector, which results in the state $\left|s^3\right\rangle = Z\left|s^2\right\rangle$, and the decision-making is conducted by the same manner as for the state $\left|s^1\right\rangle$.

Now let us present a general description of the process (Kagan & Ben-Gal, 2008), that implements the above-indicated equivalence classes A^f and A^g. As indicated above, the evolution of the quantum-mechanical system with observations does not depend on the previous states and starts from the value of the state, which is obtained by the measurement. Thus, the outputs $y \in Y$ of the system are specified by a Markov process, which is controlled by input states $x \in X$ and inner states $s \in S$. Then the probability $p(y) = p\left(A^f(y)\right)$ of the equivalence class $A^f(y) = f^{-1}(y)$ is defined as a sum

$p(y) = \sum_{(x,s) \in A^f(y)} p(x,s)$ of the probabilities of the pairs $(x,s) \in X \times S$. Similarly, the probability $p(s)$ of the inner state $s \in S$ of the system is defined by the use of equivalence class $A^g(s) = g^{-1}(s)$ as a sum $p(s) = \sum_{(x,s) \in A^g(s)} p(x,s)$.

Recall that, by definition, the equivalence classes $A^f(y)$, $y \in Y$, and A^g, $A^g(s)$, form partitions α^f and α^g of the space $X \times S$. Then, the relation between the dynamics of inner states is determined by the relation between the partitions α^f and α^g. For example, let $d_{Orn}(\alpha^g, \alpha^f) = \sum_{i=0}^{k-1} \left| p(A_i^g) - p(A_i^f) \right|$, where $k = \max\{|\alpha^g|, |\alpha^f|\}$, and if $|\alpha^g| > |\alpha^f|$, then α^f is completed by empty sets, while if $|\alpha^g| < |\alpha^f|$, then empty sets are added to α^g, being Ornstein distance between the partitions (Ornstein, 1974). Then, for the finite time case, $t = 0, 1, 2, \ldots, T-1$, the Ornstein distance between partitions α_t^f and α_t^g, is defined as $d_{Orn}\left(\{\alpha_t^g\}_0^{T-1}, \{\alpha_t^f\}_0^{T-1}\right) = (1/(T-1)) \sum_{t=0}^{T-1} d_{Orn}(\alpha_t^g, \alpha_t^f)$. Since the structure of the partitions α_t^f and α_t^g is constant, the distance represents the relation between probability measures defined by these partitions. Thus the goal of the robots navigation is to find a process for governing the inner states $s \in S$ such that the distance $d_{Orn}\left(\{\alpha_t^g\}_0^{T-1}, \{\alpha_t^f\}_0^{T-1}\right)$ reaches its minimum over the considered time interval. Below we will consider general informational algorithms of local search, which can be implemented for such a task.

5. Information theoretic decision-making and path planning

The section presents information theoretic methods for quantum inspired decision-making and general path-planning algorithms. We start with a motivating example of informational decision-making, then we consider the logic of quantum mechanics and informational distance between the partitions of the events space. Finally, we present the navigation algorithms, which are based on the representation of the states' evolution by the use of partitions.

5.1 Decision-making by the use of quantum-mechanical information measure

An application of informational criteria for decision-making and path planning of the quantum-controlled mobile robots is motivated by the criteria of classical information theory (Cover & Thomas, 1991). Recall that in the classical case, an informational distance between the probability vectors $\bar{p} = (p_1, \ldots, p_n)$ and $\bar{q} = (q_1, \ldots, q_n)$, $\sum_{j=1}^n p_j = 1$, $\sum_{j=1}^n q_j = 1$, $p_j \geq 0$, $q_j > 0$, $j = 1, 2, \ldots, n$, is specified by the *relative Shannon entropy* or *Kullback-Leibler distance* $KL(\bar{p} \,|\, \bar{q}) = \sum_{j=1}^n p_j \log(p_j/q_j) = \sum_{j=1}^n p_j \log p_j - \sum_{j=1}^n p_j \log q_j$, where by convention it is assumed that $0 \log 0 = 0$. The distance $KL(\bar{p} \,|\, \bar{q})$ satisfies both $KL(\bar{p} \,|\, \bar{q}) \geq 0$ and $KL(\bar{p} \,|\, \bar{q}) = 0$ if and only if $p_j = q_j$ for all $j = 1, 2, \ldots, n$, and, in general

$KL(\bar{p}||\bar{q}) \neq KL(\bar{q}||\bar{p})$. Vectors $\bar{p}$ and $\bar{q}$ represent the states of the stochastic system, and distance $KL(\bar{p}||\bar{q})$ characterizes the information-theoretic difference between these states.

In contrast, in quantum-mechanical systems, the state is represented by the above-presented Hermitian density matrix σ with $\mathrm{tr}(\sigma) = 1$. The informational measures for such states are defined on the basis of the von Neumann entropy (Nielsen & Chuang, 2000) $VN(\sigma) = -\mathrm{tr}(\sigma \log \sigma) = -\sum_{j=1}^{n} \lambda_j \log \lambda_j$, where λ_j are the eigenvalues of the matrix σ. Then the *relative von Neumann entropy* of the state σ' relative to the state σ'' is defined as $VN(\sigma'||\sigma'') = \mathrm{tr}(\sigma' \log \sigma') - \mathrm{tr}(\sigma' \log \sigma'') = -\sum_{j=1}^{n} \lambda_j' \log \lambda_j' - \sum_{j=1}^{n} \lambda_j' \log \lambda_j''$, where λ' and λ'' are eigenvalues of the matrices σ' and σ'', correspondingly.

Let σ^t be a state of the system at time moment t, and consider its representation $\sigma^t = \sum_{j=1}^{n} \lambda_j^t E_j$, where $\lambda_1^t < \lambda_2^t < ... < \lambda_n^t$ are eigenvalues of the matrix σ^t and $E = \{E_1, ..., E_n\}$ is a set of matrices such that $E_j E_k = \delta_{jk} E_k$ and $\sum_{j=1}^{n} E_j = I$, where $\delta_{jk} = 1$ if $j = k$, and $\delta_{jk} = 0$ otherwise. According to the dynamics of the quantum system, if the system is in the state σ^t, then its next state σ^{t+1} is specified by the use of the selected operator E_j according to the projection postulate $\sigma^{t+1} = E_j \sigma^t E_j / \mathrm{tr}(\sigma^t E_j)$. This postulate represents the above indicated influence of the measurement, i.e. of the application of the operator E_j, to the state of the system, and the generalized the Bayesian rule to the evolution of quantum-mechanical systems. The decision-making problem required a definition of such projection that given a state σ^t, the next state σ^{t+1} is optimal in a certain sense. Since for a state σ^t, there exist several sets E of matrices with the above indicated properties, the decision-making includes two stages (Davies, 1978; Holevo, 2001) and requires finding the set $E = \{E_1, ..., E_n\}$, and then selecting an operator E_j from the set E according to optimality criteria.

One of the methods, which is widely used in classical information theory, implies a choice of such probability vector $\bar{p}^{t+1}$, that given a vector $\bar{p}^t$, the Kullback-Leibler distance $KL(\bar{p}^t||\bar{p}^{t+1})$ reaches its maximum. Likewise, in the simulations, we implemented the choice of the set E and operator $E_j \in E$ such that it maximizes the relative von Neumann entropy $VN(\sigma^t||\sigma^{t+1})$. In the simulations (Kagan et al., 2008), the mission of the robot was to navigate in the environment and to find the objects, which randomly change their location. The amplitudes, which defined the states, were derived from the distances between the objects and as they were measured by the ultra-sonic sensor. The scheme of the simulation and the fragment of the robot's movement are shown in Fig. 3.

The robot scans the environment, chooses such an object that maximizes the von Neumann relative entropy, and moves to this object. After the collision, the object was moved to a new random location. The comparison between quantum and classical decision-making demonstrated the difference in nearly 50% of the decisions, and in the search for variable number of objects, quantum decision-making demonstrated nearly 10% fewer decision errors than the classical one. Such results motivated an application of information theoretic methods for navigation of quantum-controlled mobile robots. In the next section we consider the algorithms which follow this direction.

Fig. 3. The mobile robot with ultra-sonic sensor acting in the stochastic environment: after collision, the obstacle randomly changes its location in the environment.

5.2 Logic of quantum mechanics and informational distance

Now let us consider the logic of quantum mechanics and the structure of quantum events over which the informational algorithms act. Such a formal scheme of quantum events, which is called *quantum logic* (Cohen, 1989), was introduced by Birkhoff and von Neumann (Birkhoff & Neumann, 1936) as an attempt of to find an axiomatic description of quantum mechanics.

A quantum logic is a lattice Λ of events A, which contains the smallest element $\varnothing$, the greatest element I, relation $\subset$, unary operation $'$, and binary operations $\cup$ and $\cap$. It is assumed that for the events $A \in \Lambda$ the following usual set properties hold true: a) For any element $A \in \Lambda$ there exist an event $A' \in \Lambda$ such that $(A')' = A$; b) $A \cap A' = \varnothing$ and $A \cup A' = I$; c) For any pair of events $A, B \in \Lambda$, $A \subset B$ implies $B' \subset A'$, and $A \subset B$ implies $B = A \cup (B \cap A')$; d) For any countable sequence $A_1, A_2, \ldots$ of events from Λ their union is in Λ, i.e., $(\cup_j A_j) \in \Lambda$. Notice that is not required that the events $A \in \Lambda$ are subsets of the same set. That leads to the fact that under the requirements of quantum logic the distributive rule may not hold and, in general, $B \cap (A \cup A') \geq (B \cap A) \cup (B \cap A')$. In contrast, in the probabilistic scheme, it is postulated that the events are the subsets of the set of elementary outcomes, so the distribution rule is satisfied.

In quantum logic Λ, events $A \in \Lambda$ and $B \in \Lambda$ are called *orthogonal*, and denoted by $A \perp B$, if $A \subset B'$, and a finite set $\alpha = \{A_1, \ldots, A_n\}$, of events $A_j \in \Lambda$, is called $\vee$-*orthogonal system* if $\left(\cup_{j=1}^{k} A_j \right) \perp A_{k+1}$, $k = 1, 2, \ldots, n-1$. The state μ over the quantum logic Λ is defined as a map $\mu : \Lambda \to [0,1]$ such that $\mu(I) = 1$ and for any $\vee$-orthogonal system $\alpha = \{A_1, \ldots, A_n\}$ it holds true that $\mu\left(\cup_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} \mu(A_j)$. Given a state $\mu : \Lambda \to [0,1]$, an $\vee$-orthogonal system α is a *partition* of the logic Λ with respect to the state μ if $\mu\left(\cup_{j=1}^{n} A_j \right) = 1$, $A_j \in \alpha$ (Yuan, 2005).

Following classical ergodic theory (Rokhlin, 1967), *entropy* of the partition α is defined as $H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A)$, and *conditional entropy* of partition α relatively to partition β is $H_\mu(\alpha \,|\, \beta) = -\sum_{B \in \beta} \sum_{A \in \alpha} \mu(A, B) \log \mu(A \,|\, B)$, where α and β are partitions of quantum logic Λ (Yuan, 2005; Zhao & Ma, 2007). In addition, similarly to the ergodic theory (Rokhlin,

1967), a Rokhlin distance is defined as $d_\mu(\alpha, \beta) = H_\mu(\alpha | \beta) + H_\mu(\beta | \alpha)$ between the partitions α and β of the quantum logic Λ, which preserves the metric properties (Khare & Roy, 2008), as it holds for the probabilistic scheme.

5.3 Actions of the robot over the quantum events

The indicated properties of the partitions and states of quantum logic allow an application of informational A*-type search algorithms acting over partitions space (Kagan & Ben-Gal, 2006) for navigation of quantum-controlled mobile robots. Let χ be a set of all possible partitions of the logic Λ given a state μ. By $d_\mu(\alpha, \beta)$, $\alpha, \beta \in \chi$, we denote a Rokhlin distance between the partitions α and β, and by $\tilde{d}_\mu(\alpha, \beta)$ an estimated distance such that $\tilde{d}_\mu(\alpha, \beta) \le d_\mu(\alpha, \beta)$; as a distance $\tilde{d}_\mu$ the above-defined Ornstein distance $d_{Orn}(\alpha, \beta)$ can be applied. Let $r \in R^+$ be a constant value. The non-empty neighborhood $N(\alpha, r) \subset \chi$ of partition $\alpha \in \chi$ is a set of partitions such that $\alpha \notin N(\alpha, r)$ and for every partition $\beta \in N(\alpha, r)$ it holds true that $\tilde{d}_\mu(\alpha, \beta) \le r \le d_\mu(\alpha, \beta)$. In the algorithm, α stands for a partition specified by the robot, and τ stands for a partition specified by the environment. The robot is located in the environment, which is specified by a partition $\tau_{cur} = \vartheta$ and starts with the initial partition $\alpha_{cur} = \theta$. Then, the actions of the robot are defined by the following sequence:

The robots actions given the environment τ_{cur}:

-   Choose the next partition $\alpha_{next} \leftarrow \arg\min_{\alpha \in N(\alpha_{cur}, r)} \left\{ \tilde{d}_\mu(\alpha, \tau_{cur}) \right\}$;

-   Update estimation $\tilde{d}_\mu(\alpha_{cur}, \tau_{cur}) \leftarrow \max \left\{ \tilde{d}_\mu(\alpha_{cur}, \tau_{cur}), r + \min_{\alpha \in N(\alpha_{cur}, r)} \tilde{d}_\mu(\alpha, \tau_{cur}) \right\}$;

-   Set current partition $\alpha_{cur} \leftarrow \alpha_{next}$.

The robots actions while the environment changes ($\tau_{cur} \leftarrow \tau$):

-   Update estimation $\tilde{d}_\mu(\alpha_{cur}, \tau) \leftarrow \max \left\{ \tilde{d}_\mu(\alpha_{cur}, \tau), \tilde{d}_\mu(\alpha_{cur}, \tau_{cur}) - r \right\}$.

Let us clarify the application of the algorithms to the quantum logic Λ by half steps, $t = 0, 1/2, 1, 3/2, \ldots$, which correspond to the robot's and the environment's actions. Let the initial map by $\mu^0 : \Lambda \rightarrow [0, 1]$. Regarding the robot's states, it means a choice of the basis for the states; in the above-given examples, such a choice corresponds to the definition of the qubits for the states $"\uparrow" = |s_1\rangle$ and $"\downarrow" = |s_2\rangle$, or that is equivalent to the states $"\rightarrow" = |s_3\rangle$ and $"\leftarrow" = |s_4\rangle$. By the use of the map μ^0 the robot chooses the state $|s^0\rangle$ and a certain partition which specifies the next map $\mu^{1/2}$. According to the obtained map $\mu^{1/2}$, the environment changes and the map μ^1 is specified. Now the robot chooses the $|s^1\rangle$ and according to the chosen partition specifies the map $\mu^{3/2}$. The environment, in its turn, changes according to the map $\mu^{3/2}$, and so on.

The presented actions over the partitions of quantum logic provide a representation, which differs from the one use in the quantum learning algorithms (Chen & Dong, 2008; Dong, et al., 2010) and implements information theoretic decision-making using the Rokhlin distance. However, the meaning of the actions is similar, and the goal of the robot is to determine the

actions such that the inner states of the robot correspond to the environmental states, which, in their turn, are changed by the robot.

6. Notes on fuzzy logic representation of quantum control

Let us return to the representation of the robot's states by qubits $|s\rangle = a|0\rangle + b|1\rangle = a(1,0)^T + b(0,1)^T$, where both amplitudes a and b are real numbers. In such a case, the state-vector $|s\rangle$ can be represented by two membership functions $\mu_a : X \to [0,1]$ and $\mu_b : X \to [0,1]$ for some universal set X, which are defined as (Hannachi, et al., 2007)

$$\mu_a = (2/\pi)\arcsin\sqrt{\left(sign(a)a^2 - sign(b)b^2 + 1\right)/2} \text{ and } \mu_b = (2/\pi)\arcsin\sqrt{\left(sign(a)a^2 + sign(b)b^2 + 1\right)/2},$$

with backward transformations $a = sign(\mu_a + \mu_b - 1)\sqrt{\left|\sin^2 \mu_a\pi/2 + \sin^2 \mu_b\pi/2 - 1\right|}$ and

$b = sign(\mu_b - \mu_a)\sqrt{\left|\sin^2 \mu_b\pi/2 - \sin^2 \mu_a\pi/2\right|}$. Over the pairs of membership functions μ_a and μ_b, fuzzy analogs of quantum mechanical operators are defined (Hannachi, et al., 2007). Let us consider the Hadamard operators, which represent the turns of the robot.

Fuzzy analog $\tilde{H}$ of the Hadamard operator H is the following (Hannachi, et al., 2007): $\tilde{H}(\mu_a, \mu_b) = \left(\max\left(0, \min(1, 1/2 - \mu_a + \mu_b)\right), \min\left(1, \max(0, \mu_a + \mu_b - 1/2)\right)\right)$. Similarly, reverse fuzzy Hadamard operator is defined by the following transformation (Rybalov et al., 2010): $\tilde{H}^R(\mu_a, \mu_b) = \left(\min\left(1, \max(0, 3/2 - \mu_a - \mu_b)\right), \max\left(0, \min(1, 1/2 - \mu_a + \mu_b)\right)\right)$.

Straightforward calculations show that the direct and reverse fuzzy Hadamard operators $\tilde{H}$ and $\tilde{H}^R$ are reversible and preserve the properties of corresponding quantum Hadamard operators. Trajectories of the robot that acts according to the fuzzy Hadamard operators are illusrated by Fig. 4; the simulattions have been conducted likewise in the example shown in Fig. 1.
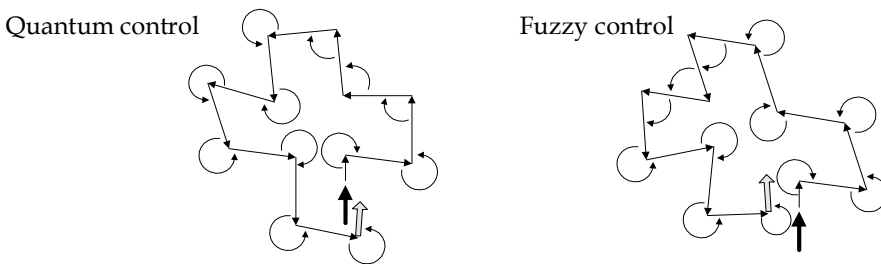

Fig. 4. Trajectories and turns of the mobile robot according to quantum and fuzzy control.

According to simulations, the turns of quantum and fuzzy controlled robot are different; however, the states of the robot and the results of its actions are statistically equivalent. Such preliminary results show that in case of real amplitudes, fuzzy logic models of quantum control may be applied.

7. Conclusion

In the chapter we presented a brief introduction into the methods of navigation of quantum-controlled mobile robots and considered the ideas of its implementation by the use of probabilistic and information theoretic techniques. The described methods represent such a property of the quantum-controlled mobile robots that the state of quantum-mechanical system includes the state of the environment as a part of its inner state.

In particular, the state of the mobile robot in the environment was defined by the use of the density matrix, which, in addition to the inner state of the robot, included the state of the environment. Such a specification of the state allowed calculations of both the robot's influence on the environment and the environmental influence on the robot by the use of the unified techniques.

The decision-making methods, which define the robot's behavior, implemented the indicated representation of the state and were based on the probabilistic and informational schemes. These schemes generalize the known maximum probability and maximum information criteria while taking into account additional information regarding the robot's influence on the environment, and correspond to the statistical considerations of quantum-mechanical methods (Malley & Hornstein, 1993; Barndorff-Nielsen & Gill, 2003).

In general, the actions of quantum-controlled mobile robot were specified by the choices of certain partitions of quantum logic. The choices were based on informational distances following the line of informational search algorithms (Kagan & Ben-Gal, 2006). As indicated above, such a method gives an alternative view to quantum learning and path-planning algorithms (Chen & Dong, 2008; Dong, et al., 2010).

The presented methods were simulated by the use of small mobile robots, while the complete realization of quantum control requires quantum-mechanical on-board computers. However, as it follows from preliminary considerations (Rybalov, et al., 2010), fuzzy control of the mobile robot demonstrates similar results as probabilistic and informational schemes of quantum control; thus in some cases fuzzy logic models of quantum control may be applied.

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