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On the uncertainties of decentralized controllers in a transfer production line

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In this paper, an information theoretic approach is applied to analyze the performance of a decentralized control system. The control system plays the role of a correcting device which decreases the uncertainties associated with state variables of a production line by applying an appropriate "correcting signal" for each deviation from the target. In particular, a distributed feedback control policy is considered to govern a transfer production line, which consists of machines and buffers and processes a single part type in response to a stochastic demand. It is shown how the uncertainty of the demand propagates dynamically into the production system, causing uncertainties associated with buffer levels and machine production rates. The paper proposes upper estimates for these uncertainties as functions of the demand variance, parameters of the distributed controllers and some physical properties of the production line. The bounds are based on dynamic entropy measures of the system state and the control variables. Some practical implications into the area of decentralized controller design are proposed, an information-economical analysis is presented and a numerical study is performed.

1. Control, cybernetics and information theory

The complexity of a system is often measured by the variety of its state variables, that is, the cardinality of its state space and the transition process from state to state. A common performance measure of production systems, at a given point in time, is the difference between the output target and the actual system output, which is often stochastic due to noise effects. Controlling such systems over a period of time means to enforce certain rules and constraints on their state variables in order to minimize the variance of this difference. An alternative related objective of the control system is to meet the output target exactly (assuming the system has enough production capacity), while minimizing the variety of the system state variables in order to decrease costs associated with state uncertainty. For example, consider a production line which consists of machines and buffers. State variables and control variables are defined, respectively, to be the buffer levels and the machine production rates. Accordingly, the line complexity is measured by the cardinality of its buffer states, and the control system complexity is measured by the cardinality of states related to machine production rates. The objective of the controllers in a pull system might be to meet the demand exactly, while maintaining a constant work-in-process in the buffers.

A well known principle in cybernetics is the *law of requisite variety* (Wiener, 1961), stating (in its simplest form) that for a system with given variety (complexity) V, one can decrease the variety of the system output, V_0, by increasing the variety of the control system V_c, as

$$V_0 \leq \frac{V}{V_c}. \tag{1}$$

This interesting and somewhat paradoxical statement points out that reduction of complexity in one system depends on increase of complexity in the other system.

Levitin (1994) further develops the above principle by using concepts of information theory. He considers a controlled system with a random state variable X and a control variable U that are under the effect of perturbation (noise). Suppose that the controller objective is to maintain X at a constant level $X = x^0$. Then, an adequate performance measure of the controller can be characterized by the *entropy* of the state variable $H(X)$. Shannon (1948) proposed the entropy as a measure of uncertainty and defined the *differential entropy* for a continuous state space as:

$$H(X) = -\int_{\{x\}} f_X(x) \log f_X(x) \mathrm{d}x, \tag{2}$$

where $f_X(x)$ is the probability density function (pdf) of random variable (r.v.) X. Unlike the discrete entropy, the differential entropy is not an intuitive measure of uncertainty and can even carry a negative value.

However, the properties of the discrete case can be maintained by using the differential entropy as a relative quantity, measuring the differences of uncertainties, as done in this paper. Levitin (1994) notes that if a perfect controller can naturalize all the noise effects, the uncertainty of the system state is equal to zero since $H(x^0) = 0$. However, in the presence of an affecting noise, Z, the state variable fluctuates and its entropy $H(X) = H(X(Z)) \neq 0$. Applying a control effort to the system can decrease the uncertainty associated with the controlled state variable $X|U = u$, which can be measured by the conditional entropy,

$$H(X|U) = -\int_{\{u\}} f_U(u)H(X|U = u)\mathrm{d}u = -\int_{\{u\}}\int_{\{x\}} f_{U,X}(u,x)\log\big(f_{X|U}(x|u)\big)\mathrm{d}x\,\mathrm{d}u, \tag{3}$$

where $f_{U,X}(u,x)$ and $f_{X|U}(x|u)$ are, respectively, the joint pdf of X and U, and the conditional pdf of X given $U = u$. It is well known (Cover and Thomas, 1991) that for any pair of random variables $H(X|U) \leq H(X)$, where equality is obtained only if X and U are mutually independent. Thus, applying a control effort to the system can not increase the uncertainty associated with the controlled state variable. The difference between these quantities is called (*mutual*) *information* (Shannon, 1948), a known positive and symmetric measure, given by

$$I(X;U) = H(X) - H(X|U) = H(U) - H(U|X) = I(U;X). \tag{4}$$

It is seen that

$$H(X|U) = H(X) - H(U) + H(U|X). \tag{5}$$

Expression (5) shows that reducing the uncertainty of the controlled state variable $H(X|U)$ is achieved by increasing the variety of the control variable measured by $H(U)$, as stated above by the *law of requisite variety* (1). Attaining the equality $H(X) = H(U)$ means that the control system can apply an appropriate correcting signal for each possible value of X whenever a deviation from the target occurs. Moreover, note from (5) that further reduction of $H(X|U)$ is achieved by decreasing $H(U|X)$, which measures the uncertainty associated with the functional relation between state and control variables. $H(U|X)$ implies the importance of ensuring maximum adequacy of the control action, i.e., that a deterministic control system generates an exact value of U which is required to correct the deviation of X. $H(U|X) = 0$ means that the state variable X uniquely determines the control variable U. In practice, one can eliminate the uncertainty effects of $H(U|X)$ by defining a feedback control policy that associates a single control action with each of the system states.

A different information-theoretic approach was used in Saridis (1988) and Valavanis and Saridis (1988). There, a probabilistic control paradigm was posed by assigning a distribution function representing the uncertainty of selecting the appropriate control law over the space of admissible controls. Entropy was used as a measure of uncertainty to develop a design procedure for intelligent control systems. The procedure was applied to the design of intelligent robotic systems, as opposed to the production lines considered here.

In this paper, we extend the ideas stated above in two directions. First, we formulate and analyze the uncertainties of control systems with time dependence. In such systems, the continuous state of the system at a certain point of time, $X(t)$, generates a control action $U(t) = U(X(t))$ by a given control policy. The control action, in turn, affects the new system state $X(t + \mathrm{d}t)$, etc. Second, we apply the above analysis into a framework of decentralized production control systems. In particular, a transfer line with deterministic decentralized controllers is considered. Each controller determines the production rate of a machine by mapping the state space of the downstream buffer into the control action space. The motivation to analyze the uncertainties in transfer production lines was stated by Gershwin (1994): "*There is an important gap in research literature. There are almost no published papers on the variability performance of transfer lines. Although the analytic decomposition is extremely accurate in predicting mean performance, it has nothing to say about the variance of the number of parts produced,.., or the variance of buffer levels*".

The paper is organized as follows. A description of a transfer production line with a decentralized control system is given in Section 2. Section 3 derives analytic expressions that represent the uncertainty associated with the system states – reflected by buffer levels, and the uncertainty associated with the system controllers – reflected by machine production rates. Such expressions enable us to analyze the system uncertainties with respect to time, demand rates and some physical properties of the production line. Section 4 includes a discussion related to allocation policy of controllers to machines, deviation from demand targets and an information-economical analysis of the tradeoff between uncertainty costs and system costs. Section 5 provides a numerical analysis of a large-scale transfer line with discrete states and discrete time index. Section 6 summarizes the paper.

2. System description

Consider a K-machine serial production system, depicted in Fig. 1, which is similar to the *transfer line* as described by Gershwin (1994). It is a linear network of reliable

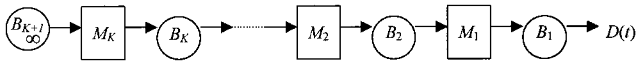

Fig. 1. The transfer line.

service-stations or machines $(M_K, M_{K-1}, \ldots, M_1)$ separated by buffer storages $(B_K, B_{K-1}, \ldots, B_2)$ with a finite capacity and equipped with an input buffer (B_{K+1}) which is an infinite source and an output buffer (B_1) with a finite capacity. The serial production system processes a single part type, which requires one operation at each machine. Buffers carry the parts between two consecutive operations.

The output buffer (B_1) is depleted at a demand rate $D(t)$, which is modeled as a stochastic process, thus, at each point of time $D(t)$ is a random variable representing the uncertainty in the demand rate. Such uncertainty is expressed by a dynamic distribution function $F_D(d,t) = \Pr[D(t) \leq d]$, giving the probability of the event that the demand rate at time t is not higher than d. Hence, the probability density function of $D(t)$ is defined as

$$f_D(d,t) = \frac{\partial}{\partial d} F_D(d,t). \tag{6}$$

Denote the kth buffer state at time t by $X_k(t)$ where $X_k(t) = \{-\infty, x_k^{\max}\}$. Then, the dynamics of the production process can be described by the following differential equations:

$$\begin{aligned} \dot{X}_1(t) &= U_1(t) - D(t), \quad X_1(0) = x_1^0, \\ \dot{X}_k(t) &= U_k(t) - U_{k-1}(t), \quad X_k(0) = x_k^0,\ k = 2, \ldots, K, \end{aligned} \tag{7}$$

where $\dot{X}_k(t)$ is the rate of increase or decrease of the kth buffer level; x_k^0 is the initial level of the corresponding buffer; and $U_k(t)$ is the controllable production rate of the corresponding machine. It is assumed that the production rates can be set arbitrarily within the capacity limits of the machines, i.e.,

$$0 \leq U_k(t) \leq u_k^{\max}. \tag{8}$$

The described production system can be controlled by applying different control policies. In this paper, we are interested in decentralized control paradigms and consider distributed proportional feedback controllers. Such a form of the control is selected to facilitate the derivation of analytical expressions from differential equations (7) which become linear. In particular, we consider the following form:

$$U_k(X_k(t)) = \begin{cases} u_k^{\max}, & \text{if } X_k(t) < x_k^{\max} - \dfrac{u_k^{\max}}{G_k} \text{ and } X_{k+1}(t) > 0, \\ G_k(x_k^{\max} - X_k(t)), & \text{if } x_k^{\max} - \dfrac{u_k^{\max}}{G_k} \leq X_k(t) \leq x_k^{\max} \text{ and } X_{k+1}(t) > 0, \\ 0, & \text{if } X_k(t) > x_k^{\max} \text{ or } X_{k+1}(t) = 0. \end{cases} \tag{9}$$

Thus, if the previous $(k+1)$th buffer is not empty, the kth machine operates according to the following rules: (i) when the current buffer level $X_k(t)$ is lower than a certain point (first line in (9)), the machine produces at the full capacity (full production regime); (ii) when, on the contrary, the current buffer level $X_k(t)$ is higher than another certain point (third line in (9)), the machine is idle (no-production regime); finally, (iii) when the current buffer level $X_k(t)$ is intermediate (second line in (9)), the machine produces at an intermediate rate proportional to the current deviation from the maximum buffer level (proportional production regime). The proportionality coefficient G_k reflects sensitivity of the control action to the deviation. For example, when $G_k = u_k^{\max}/x_k^{\max}$ the production rate decreases in a linear rate within the range $X_k(t) \in (0,\ x_k^{\max})$.

3. Uncertainty measures of the system

Let us consider the case of: (i) equal production capacities of the machines, $u_k^{\max} = u^{\max}\,\forall\, k$; (ii) identical buffer capacities, $x_k^{\max} = x^{\max}\,\forall\, k$; and (iii) identical proportionality coefficients $G_k = u^{\max}/x^{\max}\,\forall\, k$. Then, the proportional production turns out to be the unique regime along the production run time. When the kth buffer level reaches the maximum value $x^{\max}$, the production at the kth machine stops and $X_k(t)$ depletes according to (7), therefore, the no-production regime never occurs. On the other hand, when $X_k(t)$ drops to zero, $U_k(t) = u^{\max}$ and since $U_{k-1}(t) \leq u^{\max}$, we conclude from (7) that $\dot{X}_k(t) \geq 0$, meaning that the full production regime never occurs as well. The dynamic equations for the buffers at the proportional production regime take the following form by applying (9) to (7):

$$\begin{aligned} \dot{X}_1(t) &= G(x^{\max} - X_1(t)) - D(t), \quad X_1(0) = 0, \\ \dot{X}_k(t) &= G(x^{\max} - X_k(t)) - U_{k-1}(t), \\ X_k(0) &= 0, \quad k = 2, \ldots, K. \end{aligned} \tag{10}$$

The solution of these equations is:

$$\begin{aligned} X_1(t) &= x^{\max}\left(1 - \mathrm{e}^{-Gt}\right) - \mathrm{e}^{-Gt} \int_0^t \mathrm{e}^{G\tau} D(\tau)\mathrm{d}\tau, \\ X_k(t) &= x^{\max}\left(1 - \mathrm{e}^{-Gt}\right) - \mathrm{e}^{-Gt} \int_0^t \mathrm{e}^{G\tau} U_{k-1}(\tau)\mathrm{d}\tau. \end{aligned} \tag{11}$$

To analyze the statistics of the state and control variables, we assume that $D(t)$ is a stationary stochastic process with expected value $E\{D(t)\} = u^{\max}$ and autocovariance $\text{Cov}\{D\}(t_1, t_2) = \sigma^2$. Thus, the demand is modeled by an identical random variable at any point of time, $D(t) = D(t')\,\forall\, t, t'$. In many cases, the autocovariance of a stochastic process is given by

$$\text{Cov}\{D\}(t_1, t_2) = \sigma^2 e^{-c|t_1 - t_2|}.$$

For these cases, it is assumed that the exponent constant $c \to 0$, or that the production run time is such that $|t_1 - t_2| \ll 1/c$.

Lemma 1. *Let $D(t)$ be a stationary stochastic process with expected value $E\{D(t)\} = \mu$ and autocovariance $\text{Cov}\{D\}(t_1, t_2) = \sigma^2$. Then, the distribution parameters of the buffer states and control efforts are given by*

$$E\{X_k(t)\} = \left(x^{\max} - \frac{\mu}{G}\right) e^{-Gt} F_k(Gt),$$
$$E\{U_k(t)\} = u^{\max} - (u^{\max} - \mu) e^{-Gt} F_k(Gt), \tag{12}$$

and

$$\text{Cov}\{X_k\}(t_1, t_2) = \left[\frac{\sigma}{G}\right]^2 e^{-G(t_1+t_2)} F_k(Gt_1) F_k(Gt_2);$$
$$\text{Var}\{X_k(t)\} = \left[\frac{\sigma e^{-Gt} F_k(Gt)}{G}\right]^2,$$
$$\text{Cov}\{U_k\}(t_1, t_2) = \sigma^2 e^{-G(t_1+t_2)} F_k(Gt_1) F_k(Gt_2);$$
$$\text{Var}\{U_k(t)\} = \left[\sigma e^{-Gt} F_k(Gt)\right]^2, \tag{13}$$

where functions

$$F_k(z) = \sum_{i=k}^{\infty} \frac{z^i}{i!}, \quad k = 0, 1, \ldots, \infty.$$

That is, $F_k(z)$ is a Taylor expansion of e^z dropping the first k terms.

Proof. The proof is by induction. Initially, we determine the statistics of the first machine variables. From (11) we find:

$$E\{X_1(t)\} = x^{\max}\left(1 - e^{-Gt}\right) - e^{-Gt} E\left\{\int_0^t e^{G\tau} D(\tau) d\tau\right\}$$
$$= x^{\max}\left(1 - e^{-Gt}\right) - \frac{e^{-Gt}(e^{Gt} - 1)\mu}{G}$$
$$= \left(x^{\max} - \frac{\mu}{G}\right) e^{-Gt} F_1(Gt), \tag{14}$$

and

$$\text{Cov}\{X_1\}(t_1, t_2) = e^{-G(t_1+t_2)} \text{Cov}\left\{\int_0^t e^{Gt} D(\tau) d\tau\right\}$$
$$= e^{-G(t_1+t_2)} \int_0^{t_1}\int_0^{t_2} e^{G(\tau_1+\tau_2)} \text{Cov}\{D\}(\tau_1, \tau_2) d\tau_1\, d\tau_2$$
$$= \left[\frac{\sigma}{G}\right]^2 e^{-G(t_1+t_2)} F_1(Gt_1) F_1(Gt_2). \tag{15}$$

By equating t_1 and t_2 in the last expression, we obtain the variance of $X_1(t)$ at point t:

$$\text{Var}\{X_1(t)\} = \left[\frac{\sigma e^{-Gt} F_1(Gt)}{G}\right]^2. \tag{16}$$

Having determined the statistics of $X_1(t)$, we find the statistics of $U_1(t)$ from the proportional production regime of the feedback control policy (9):

$$E\{U_1(t)\} = Gx^{\max} - GE\{X_1(t)\}$$
$$= u^{\max} - (u^{\max} - \mu) e^{-Gt} F_1(Gt),$$
$$\text{Cov}\{U_1\}(t_1, t_2) = G^2 \text{Cov}\{X_1\}(t_1, t_2)$$
$$= \sigma^2 e^{-G(t_1+t_2)} F_1(Gt_1) F_1(Gt_2),$$
$$\text{Var}\{U_1(t)\} = G^2 \text{Var}\{X_1(t)\} = \left[\sigma e^{-Gt} F_1(Gt)\right]^2. \tag{17}$$

By using the induction hypothesis that the lemma is true for a $(k-1)$th production stage, we derive the statistics of the kth stage as follows.

$$E\{X_k(t)\} = x^{\max}\left(1 - e^{-Gt}\right) - e^{-Gt} E\left\{\int_0^t e^{G\tau} U_{k-1}(\tau) d\tau\right\}$$
$$= x^{\max}\left(1 - e^{-Gt}\right) - e^{-Gt}$$
$$\times \int_0^t \left(e^{G\tau} Gx^{\max} - (Gx^{\max} - \mu) F_{k-1}(G\tau)\right) d\tau$$
$$= (Gx^{\max} - \mu) e^{-Gt} \int_0^t F_{k-1}(G\tau) d\tau$$
$$= \left(x^{\max} - \frac{\mu}{G}\right) e^{-Gt} F_k(Gt), \tag{18}$$

and

$$\text{Cov}\{X_k\}(t_1, t_2)$$
$$= e^{-G(t_1+t_2)} \int_0^{t_1}\int_0^{t_2} e^{G(\tau_1+\tau_2)} \text{Cov}\{U_{k-1}\}(\tau_1, \tau_2) d\tau_1\, d\tau_2$$
$$= e^{-G(t_1+t_2)} \int_0^{t_1}\int_0^{t_2} \sigma^2 F_{k-1}(G\tau_1) F_{k-1}(G\tau_2) d\tau_1\, d\tau_2$$
$$= \left[\frac{\sigma}{G}\right]^2 e^{-G(t_1+t_2)} F_k(Gt_1) F_k(Gt_2). \tag{19}$$

By equating t_1 and t_2 in the last expression, we obtain the variance of $X_k(t)$ at point t:

$$\mathrm{Var}\{X_k(t)\} = \left[\frac{\sigma \mathrm{e}^{-Gt} F_k(Gt)}{G}\right]^2. \qquad (20)$$

Having determined the statistics of $X_k(t)$, we find the statistics of $U_k(t)$ from the feedback control policy (9):

$$\begin{aligned} E\{U_k(t)\} &= Gx^{\max} - GE\{X_k(t)\} \\ &= u^{\max} - (u^{\max} - \mu)\mathrm{e}^{-Gt} F_k(Gt), \\ \mathrm{Cov}\{U_k\}(t_1, t_2) &= G^2 \mathrm{Cov}\{X_k\}(t_1, t_2) \\ &= \sigma^2 \mathrm{e}^{-G(t_1+t_2)} F_k(Gt_1) F_k(Gt_2), \\ \mathrm{Var}\{U_k(t)\} &= G^2 \mathrm{Var}\{X_k(t)\} = \left[\sigma \mathrm{e}^{-Gt} F_k(Gt)\right]^2. \qquad (21) \blacksquare \end{aligned}$$

In order to find exact analytical expressions for the uncertainty associated with the state of the system and the control efforts, one needs to determine the probability distribution of $X_k(t)$ and $U_k(t)$. However, since $X_k(t)$ are integrals of stochastic processes, which are assumed to exist in the Riemann sense for every integrable realization of the demand process, the determination of the distribution function of $X_k(t)$ is in general "hopelessly complicated", as stated by Papoulis (1991). For this reason, we determine an upper bound on the uncertainty associated with the system states and controllers by using only the first two moments of $X_k(t)$ and $U_k(t)$ as found in Lemma 1. To do so, two additional lemmas are required.

Lemma 2. *Of all distributions with the same variance, the normal distribution maximizes the entropy.*

Proof. The proof can be found in Cover and Thomas (1991). ■

It follows from Lemma 2 that the entropy of the normal distribution gives an upper bound on the uncertainty associated with a random variable in terms of its variance.

Lemma 3. *Let Λ be a normal r.v. with pdf $f_\Lambda(\lambda)$ and variance σ^2. Then the differential entropy of Λ is given by the expression*

$$H[\Lambda] = -\int f_\Lambda(\lambda) \log f_\Lambda(\lambda) \mathrm{d}\lambda = \frac{1}{2} \log 2\pi \mathrm{e} \sigma^2 \text{ bits.} \qquad (22)$$

Proof. The proof can be found in Cover and Thomas (1991). ■

Now, the main theorem can be obtained.

Theorem 1. *The uncertainty associated with the kth buffer state at time t, for the system considered above, is upper-bounded by the following expression*:

$$\tilde{H}[X_k(t)] = \frac{1}{2}\log[2\pi \mathrm{e}] + \log\left[\frac{\sigma \mathrm{e}^{-Gt} F_k(Gt)}{G}\right]. \qquad (23)$$

Similarly, the uncertainty associated with the kth controller (the controller of the kth machine) at time t is upper-bounded by the following expression

$$\tilde{H}[U_k(t)] = \frac{1}{2}\log[2\pi \mathrm{e}] + \log\left[\sigma \mathrm{e}^{-Gt} F_k(Gt)\right]. \qquad (24)$$

Proof. The proof is immediately obtained by applying Lemma 2 and Lemma 3 to Lemma 1. ■

4. Discussion

Theorem 1 enables us to express the uncertainty associated with the system states and decentralized controllers as a function of time, demand variance and location along the transfer line. Moreover, it lights an important phenomenon related to the design of decentralized controllers in transfer lines. Note from (24) that prior to the steady state, uncertainty decreases with the machine index. This is due to the fact that the logarithm is a monotone function and that $F_k(t)$ decreases with k for a given t, as exemplified in Fig. 2 for a 10 machine transfer line. This phenomenon derives a general design principle in the area of decentralized control, as follows.

Observation 1. Consider a transfer production line of the type described above and a set of machine controllers with different complexities. Then, it is reasonable to sort the controllers by their complexity and apply them in a descending order starting with the most complex controller at the first (most downstream) machine and the less complex controller at the last (most upstream) machine. Such an allocation allows a more accurate control of the downstream machines, which behavior is more stochastic.

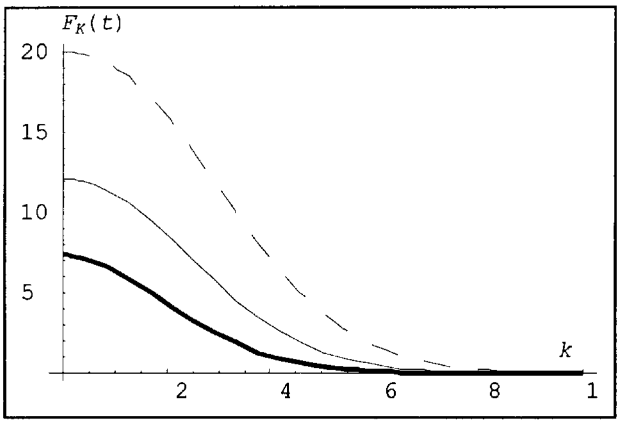

Fig. 2. Behavior of $F_k(t)$ as a function of k for $t = 2$ (bold line), $t = 2.5$ (thin line) and $t = 3$ (dashed line).

A similar conclusion can be stated in terms of buffer capacities. That is, in general, downstream buffers should have a larger capacity than upstream buffers.

Observation 2. Assume that the production line has enough capacity with respect to the expected demand, i.e., $\mu < u^{\max}$. Then, it is of interest to measure the variability (and the related uncertainty) of the difference between production and demand rates as a function of time, measured by $\mathrm{Var}\{U_1(t) - D(t)\}$. Decrease of the variance (uncertainty), leads to a more stable and more predictable process of delivering the goods to the customers. Low variance (uncertainty) of the difference between the production rates of adjacent machines, measured by $\mathrm{Var}\{U_k(t) - U_{k-1}(t)\}$, is also an important measure from reasons of line balancing and production stability. In the next lemma we calculate the above-mentioned measures on the basis of the results presented in Lemma 1.

Lemma 4.

$$\mathrm{Var}\{U_1(t) - D(t)\} = \sigma^2\left(2 - \mathrm{e}^{-Gt}\right)^2,$$
$$\mathrm{Var}\{U_k(t) - U_{k-1}(t)\} = \sigma^2\left(\mathrm{e}^{-Gt}(F_k(Gt) + F_{k-1}(Gt))\right)^2, \quad k = 2, \ldots, K. \tag{25}$$

Proof. From Lemma 1 and the system dynamics, it follows that: (i) for the last production stage

$$\begin{aligned}\mathrm{Var}\{U_1(t) - D(t)\} &= \mathrm{Var}\{U_1(t)\} + \mathrm{Var}\{D(t)\} + 2\mathrm{Cov}(U_1(t), D(t)) \\ &= \sigma^2\left[\mathrm{e}^{-Gt}F_1(Gt)\right]^2 + \sigma^2 \\ &= \sigma^2\left[1 + \left(\mathrm{e}^{-Gt}F_1(Gt)\right)^2\right].\end{aligned} \tag{26}$$

(ii) for the intermediate production stages

$$\begin{aligned}&\mathrm{Var}\{U_k(t) - U_{k-1}(t)\} \\ &= \mathrm{Var}\{U_k(t)\} + \mathrm{Var}\{U_{k-1}(t)\} + 2\mathrm{Cov}(U_k(t), U_{k-1}(t)) \\ &= \sigma^2\left[\mathrm{e}^{-Gt}F_k(Gt)\right]^2 + \sigma^2\left[\mathrm{e}^{-Gt}F_{k-1}(Gt)\right]^2 \\ &= \sigma^2\mathrm{e}^{-2Gt}\left[(F_k(Gt))^2 + (F_{k-1}(Gt))^2\right].\end{aligned} \tag{27}$$ ■

Theorem 2. *The uncertainty of the difference between the production and demand rates as well as the uncertainty between the production rates of adjacent machines are upper-bounded by*:

$$\tilde{H}[U_1(t) - D(t)] = \frac{1}{2}\log[2\pi\mathrm{e}] + \log\left[\sigma\sqrt{1 + \left(\mathrm{e}^{-Gt}F_1(Gt)\right)^2}\right],$$
$$\tilde{H}[U_k(t) - U_{k-1}(t)] = \frac{1}{2}\log[2\pi\mathrm{e}] + \log\left[\sigma\mathrm{e}^{-Gt}\sqrt{(F_k(Gt))^2 + (F_{k-1}(Gt))^2}\right]. \tag{28}$$

Proof. The proof is immediately obtained from Lemmas 2, 3 and 4. ■

Observation 3. This observation concerns the determination of maximum production capacity, $u^{\max}$, that minimizes an economic criterion for the steady state to which the system asymptotically converges. Note from (12) and (13) that when t tends to infinity, the mean and the variance of the work-in-process in the kth buffer tend to their limit values given by

$$\lim_{t\to\infty} E\{X_k(t)\} = x^{\max}\left(1 - \frac{\mu}{u^{\max}}\right);$$
$$\lim_{t\to\infty} \mathrm{Var}\{X_k(t)\} = \frac{\sigma^2}{G^2}, \quad \text{for all } k. \tag{29}$$

Let the economic criterion include three costs: carrying inventory, deviation of inventory and production capacity, i.e.

$$J(u^{\max}) = \lim_{t\to\infty}\left[p_1\sum_k E\{X_k(t)\} + p_2\sum_k \mathrm{Var}\{X_k(t)\}\right] + p_3u^{\max}, \tag{30}$$

where p_1, p_2, and p_3 are, respectively, the coefficients of the above mentioned costs (in this case, we associate cost terms directly with the variance instead of entropy since one is linearly dependent on the other).

By taking into account (29), the limit form of the criterion is given by

$$J(u^{\max}) = p_1Kx^{\max}\left(1 - \frac{\mu}{u^{\max}}\right) + p_2K\sigma^2\left(\frac{x^{\max}}{u^{\max}}\right)^2 + p_3u^{\max}. \tag{31}$$

Since the second derivative of $J(u^{\max})$ changes sign only once (from positive to negative) and since $J(u^{\max})$ tends to a linear function for $u^{\max} \to \infty$, it has a unique extremum (minimum) point. The minimum is found by using the first derivative:

$$\frac{\partial J(u^{\max})}{\partial u^{\max}} = p_1K\mu\frac{x^{\max}}{(u^{\max})^2} - 2p_2K\sigma^2\frac{(x^{\max})^2}{(u^{\max})^3} + p_3 = 0. \tag{32}$$

Solving (32) and taking into account the capacity constraint, $u^{\max} > \mu$, we obtain the optimal value of the production capacity, denoted by $\hat{u}^{\max}$, as

$$\hat{u}^{\max} = \max\left\{\mu, \left(\frac{KC}{9}\right)^{1/3}\left(\frac{x^{\max}}{p_3}\right)^{1/2} - \mu p_1\left(\frac{K^2}{3C}\right)^{1/3}\left(\frac{x^{\max}}{p_3}\right)^{1/2}\right\}, \tag{33}$$

where

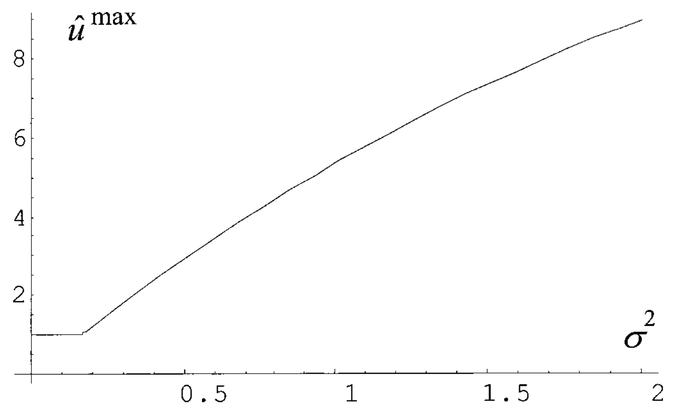

Fig. 3. Optimal production capacity as a function of demand variance for $p_1 = 2,\ p_2 = 0.5,\ p_3 = 3,\ k = 30,\ x^{\max} = 12$ and $\mu = 1$.

$$C = 9p_2\sigma^2\sqrt{p_3x^{\max}} + \sqrt{3\left(Kp_1^3\mu^3 + 27p_2^2p_3x^{\max}\sigma^4\right)}.$$

It is seen that $\hat{u}^{\max}$ increases monotonously as a function of σ^2 as exemplified by Fig. 3. Note that for a large value of σ^2, one obtains $\hat{u}^{\max}(\sigma^2) = O(\sigma^{2/3})$.

5. Numerical example of a discrete transfer line

In the following, we present a numerical study of a transfer line that operates under the suggested control policy with discrete state space and discrete time index. It is of interest to examine whether the general behavior of the continuous transfer line, as obtained analytically, is similar for this case.

The simulation model is written using the Siman V simulation language. The model consists of 100 machines separated by buffers with an infinite source and an output buffer depleted at a random demand rate. The model is executed for a horizon of 100 000 time units with a warm up period of 5000 time units (steady state is obtained for various simulation-runs with different seeds).

The deterministic machine controllers operate using a proportional feedback control paradigm, as presented in (9), with the following parameters: $u_k^{\max} = 1/2$[parts/time unit], $x_k^{\max} = 20$[parts], $G_k = 1/40$ [1/time unit], $\forall k = 1, \ldots, 100$. In the discrete case, each time that a part arrives to a machine, the machines' production rate is updated based on the discrete number of parts in the downstream buffer. The machines' production rate remains fixed for the whole production interval, thus, until a new part arrives. The initial buffer levels are set to 10 parts and the demand rate is distributed exponentially with mean of 1/3[parts/time unit]. Table 1 summarizes the statistics of more than 65 500 data points in different buffers along the line. It presents the distribution of parts in these buffers at steady state, the sample parameters, the discrete entropies and the continuous entropies. Unlike the continuous transfer line, the distribution of parts (and accordingly the buffer-level variance) in the discrete case depends on the machine index not only at transition time but also at steady state. Such phenomenon implies more importance of Observation 1 for the discrete transfer line. The discrete entropy is a measure of the buffer level uncertainty. It is calculated by $-\sum_i \delta_i \log \delta_i$, where δ_i is the frequency of the ith buffer level. The continuous entropy

Table 1. Parts distribution in different buffers at steady state, the sample parameters, the discrete and the continuous entropies

	Buffer level	*$X_1(\infty)$ (%)*	*$X_{10}(\infty)$ (%)*	*$X_{50}(\infty)$ (%)*	*$X_{90}(\infty)$ (%)*	*$X_{100}(\infty)$ (%)*
	Empty	1.140				
	1 part	3.377	–	–	–	–
	2 parts	5.808	–	–	–	–
	3 parts	8.858	0.32	–	–	–
	4 parts	11.957	5.01	–	–	–
	5 parts	14.345	21.70	17.21	16.99	16.85
	6 parts	15.088	36.60	49.05	49.79	49.94
	7 parts	13.738	27.22	32.79	33.01	33.15
	8 parts	10.857	8.38	0.95	0.21	0.06
	9 parts	7.465	0.76	–	–	–
	10 parts	4.366	–	–	–	–
	11 parts	2.034	–	–	–	–
	12 parts	0.752	–	–	–	–
	13 parts	0.182	–	–	–	–
	14 parts	0.032	–	–	–	–
	15 parts	0.003	–	–	–	–
Sample mean		5.798	6.135	6.174	6.164	6.164
Sample variance		6.395	1.103	0.507	0.481	0.475
Discrete entropy		3.372	2.116	1.532	1.482	1.467
Continuous entropy (Normal dist.)		3.385	2.118	1.557	1.519	1.510

Table 2. Square error fit of different continuous distributions to parts distribution in the first buffer at steady state

Distribution functions	*Square error w.r.t.* $X_1(\infty)$
Normal	0.0673
Beta	0.068
Triangular	0.0707
Erlang	0.0719
Gamma	0.0729
Weibull	0.0791
Uniform	0.0837
Lognormal	0.0843
Exponential	0.089

is calculated by applying (22), i.e., by using the entropy upper bound based on a normal distribution with the indicated sample variance. Note from Table 1 that the continuous entropy gives a good estimate for the discrete entropy.

The normal distribution is selected by the software as the best continuous distribution that fits the data of downstream buffers. The software sorts various applicable distributions, from best to worst, based upon the values of the respective square errors and selects the best fit. Table 2 presents the values of the square errors of various distributions with respect to the first buffer ($k = 1$). The histogram plot of this buffer is presented in Fig. 4. These observations affirm the use of the normal distribution to obtain the uncertainty upper bound, as done in previous sections.

6. Conclusions

In this paper, the uncertainties associated with a transfer production line, caused by a stochastic demand, have been studied. A decentralized feedback control has been used to govern the flow of a single part type through machines and buffers making up the line. Upper estimates of the uncertainties in the system have been found and a general rule for distributed controller design has been suggested. A natural extension to this work is the uncertainty analysis of different production systems and different control policies. This includes the investigation of the effects of machine starvation on the dynamic propagation of uncertainties into the production line. Not only the proportional feedback control paradigm, but also more complex feedback control policies (such as the bang/bang, polynomial, sigmoid etc) can be investigated by the proposed methodology. The difference between uncertainty measures of continuous and discrete transfer lines can also be studied.

A different research direction is the investigation of control "centralization effects" on the resulting uncertainties in the system. For example, one can think about two controllers, where $\bar{U}_k(t) = \bar{U}_k(X_{\bar{m}}(t), X_{\bar{m}+1}(t), \ldots, X_{\bar{n}-1}(t), X_{\bar{n}}(t))$, $\bar{m} \le k \le \bar{n}$, is considered more "centralized" than controller $\hat{U}_k(t) = \hat{U}_k(X_{\hat{m}}(t), X_{\hat{m}+1}(t), \ldots, X_{\hat{n}-1}(t), X_{\hat{n}}(t))$, $\hat{m} \le k \le \hat{n}$, if $\bar{m} \le \hat{m}$ and $\bar{n} \ge \hat{n}$. Then, the best "degree of centralization" can be determined for a specific production line as the set of controllers which minimizes the upper bound of the total uncertainty in the system, $\tilde{H}(U_k(t))$, as obtained in Theorem 1.

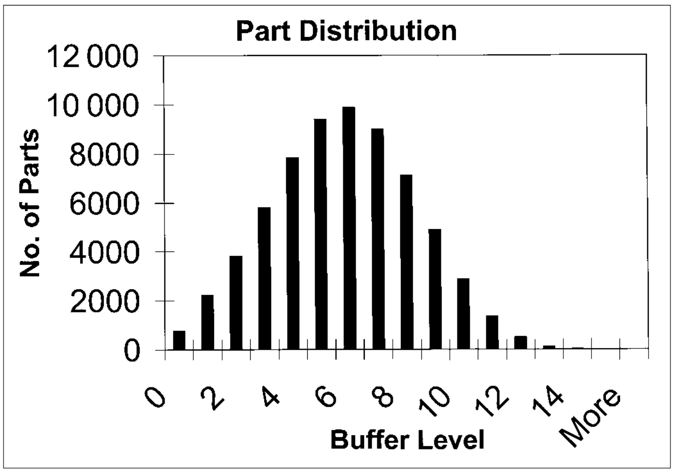

Fig. 4. Distribution of parts in the first buffer at steady-state.

References

Cover, T.M. and Thomas, J.A. (1991) *Elements of Information Theory*, John Wiley and Sons, New York, NY.

Gershwin, S.B. (1994) *Manufacturing System Engineering*, Prentice Hall, Englewood Cliffs, NJ.

Levitin, L.B. (1994) Fundamental limits of control, in *Proceedings of the First Symposium on High-Speed Civil Transportation Vehicle*, Greensboro, NC.

Papoulis, A. (1991) *Probability, Random Variables, and Stochastic Processes*, Prentice Hall, Englewood Cliffs, NJ.

Saridis, G.N. (1988) Entropy formulation of optimal and adaptive control. *IEEE Transactions on Automatic Control*, **33**, 713–721.

Shannon, C.E. (1948) The Mathematical Theory of Communication. *Bell System Technical Journal*, **27**, 379–423, 623–656.

Valavanis, K.P. and Saridis, G.N. (1988) Information-theoretic modeling of intelligent robotic systems. *IEEE Transactions on Systems, Man, and Cybernetics*, **18**, 852–872.

Wiener, N. (1961) *Cybernetics, or Control and Communication in the Animal and the Machine*, MIT. Press, Cambridge, MA.

Biographies

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